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Norms of Some Projections on $C[a, b]$

WILLIAM A. LIGHT

Department of Mathematics, University of Lancaster, Lancaster, England

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1. INTRODUCTION

Denote by $C[a, b]$ the Banach space of all continuous real-valued functions defined on $[a, b]$, with the supremum norm $\| f \| = \max_{x \in [a,b]} | f(x) |$. Denote by $P_n[a, b]$ the subspace of $C[a, b]$ consisting of all polynomials of degree at most n. Any bounded linear operator $L : C[a, b] \to P_n[a, b]$ such that $Lp = p \ \forall p \in P_n$ is a projection of $C[a, b]$ onto $P_n[a, b]$. Let F be a family of projections from $C[a, b]$ to $P_n[a, b]$. Then we say $L_0 \in F$ is minimal in F if $\| L_0 \| \leqslant \| L \| \ \forall L \in F$. Cheney and Price [1] provide an excellent survey of the whole field of projections. If the family F comprises all projections $L : C[a, b] \to P_n[a, b]$ then the minimal $\| L \|$ is not known. We shall consider a particular family of projections F defined by

$$(L_n^\alpha f)(x) = \sum_{i=0}^{n} a_i C_i^\alpha(x), \qquad \alpha > -\tfrac{1}{2},$$

where the $C_i^\alpha(x)$ are polynomials orthogonal on $[-1, 1]$ with respect to the weight function $(1 - x^2)^{\alpha - \frac{1}{2}}$ and

$$a_i = \gamma_i \int_{-1}^{1} (1 - t^2)^{\alpha - \frac{1}{2}} f(t)\, C_i^\alpha(t)\, dt,$$

where γ_i is a suitable normalization factor. These polynomials are, of course, the ultraspherical or Gegenbauer polynomials, and the case $\alpha = 0$ gives the Chebyshev polynomials $T_k(x) = \cos k\theta$, $x = \cos \theta$. It is well known [2] that

$$\| L_n^0 \| = \frac{1}{\pi} \int_0^\pi \left| \frac{\sin(n + \frac{1}{2})\, \theta}{\sin \theta/2} \right| d\theta = \frac{1}{\pi} \int_0^\pi | D_n(\theta) |\, d\theta,$$

where $D_n(\theta)$ is the Dirichlet kernel. Furthermore, by considering the function on $[0, \pi]$ given by

$$\hat{f}_n(\theta) = \operatorname{sgn}[D_n(\theta)]$$

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we obtain $\| L_n^0 \hat{f}_n \| = \| L_n^0 \|$. By reference to Zygmund [3, p. 73] we see that $\hat{f}_n(x)$ for $x \in [-1, 1]$ has a Chebyshev series expansion uniformly convergent in the intervals of continuity of $\hat{f}_n$ with Chebyshev coefficients defined as

$$d_k^{(n)} = \frac{2}{\pi k} \tan\left(\frac{\pi k}{2n+1}\right).$$

The norm of L_n^0 is now given by

$$\| L_n^0 \| = \sum_{k=0}^{n} d_k^{(n)} T_k(1) = \sum_{k=0}^{n} d_k^{(n)}.$$

We shall show that for $0 < \alpha < 1$, $(L_n^\alpha \hat{f}_n)(1)$ is well defined and $(L_n^\alpha \hat{f}_n)(1) > (L_n^0 \hat{f}_n)(1)$. We make use of a formula connecting ultraspherical polynomials for varying α, which is due to Gegenbaueur [5]. For a summary of these and other results concerning connections between families of orthogonal polynomials the recent book by Askey [6] is well worth consulting.

Let $C_n^\lambda(x) = \sum_{r=0}^{[n/2]} b_r^{(n)} C_{n-2r}^\alpha(x)$. Then the $b_r^{(n)}$ are given by

$$b_r^{(n)} = \frac{\Gamma(\alpha)(n-2r+\alpha)\,\Gamma(r+\alpha-\lambda)\,\Gamma(n-r+\lambda)}{\Gamma(\lambda)\,\Gamma(\lambda-\alpha)\,r!\,\Gamma(n-r+\alpha+1)}.$$

Now $T_n(x)$ is given by

$$T_n(x) = \frac{n}{2} \lim_{\lambda \to 0} \frac{1}{\lambda} C_n^\lambda(x).$$

Hence,

$$T_n(x) = \frac{n}{2} \lim_{\lambda \to 0} \sum_{r=0}^{[n/2]} \frac{b_r^{(n)}}{\lambda} C_{n-2r}^\alpha(x)$$

and

$$\frac{b_r^{(n)}}{\lambda} = \frac{\Gamma(\alpha)(n-2r+\alpha)\,\Gamma(r+\lambda-\alpha)\,\Gamma(n-r+\lambda)}{\lambda\Gamma(\lambda)\,\Gamma(\lambda-\alpha)\,r!\,\Gamma(n-r+\alpha+1)}$$

$$= \frac{\Gamma(\alpha)(n-2r+\alpha)\,\Gamma(r+\lambda-\alpha)\,\Gamma(n-r+\lambda)}{\Gamma(\lambda+1)\,\Gamma(\lambda-\alpha)\,r!\,\Gamma(n-r+\alpha+1)}$$

Let

$$e_r^{(n)} = \frac{n}{2} \lim_{\lambda \to 0} \frac{b_r^{(n)}}{\lambda}$$

then

$$T_n(x) = \sum_{r=0}^{[n/2]} e_r^{(n)} C_{n-2r}^\alpha(x),$$

where

$$e_r^{(n)} = \frac{n\Gamma(\alpha)(n - 2r + \alpha)\, \Gamma(r - \alpha)\, \Gamma(n - r)}{2\Gamma(-\alpha)\, r!\, \Gamma(n - r + \alpha + 1)}.$$

With the above definitions we have the following result:

LEMMA 1. *We have* $e_r^{(n)} < 0$ *and*

$$| e_r^{(n)} | > | e_{r+1}^{(n+2)} | \qquad \textit{for} \quad 1 \leqslant r \leqslant [n/2] \quad \textit{and} \quad 0 < \alpha < 1.$$

Proof. The fact that $e_r^{(n)} < 0$ for $0 < \alpha < 1$, $1 \leqslant r \leqslant [n/2]$ is obtained immediately from the observation that all the arguments to the gamma functions are positive, as are the factors $(n - 2r + \alpha)$, $r!$ except for $\Gamma(-\alpha)$, which is negative for $0 < \alpha < 1$.

The $e_r^{(n)}$, $e_{r+1}^{(n+2)}$ are simply the coefficients of C_{n-2r} in the expansions of T_n and T_{n+2}, respectively. Since they are both negative for $1 < r < [n/2]$ we have

$$\frac{| e_{r+1}^{(n+2)} |}{| e_r^{(n)} |} = \frac{e_{r+1}^{(n+2)}}{e_r^{(n)}} = \frac{(n + 2)\, \Gamma(r + 1 - \alpha)\, \Gamma(n - r - 1)}{(r + 1)!\, \Gamma(n - r + \alpha + 2)} \cdot \frac{r!\, \Gamma(n - r + \alpha + 1)}{n\Gamma(r - \alpha)\, \Gamma(n - r)}.$$

Using the relationship $\Gamma(z + 1) = z\Gamma(z)$ to eliminate the Γ terms we have

$$\frac{e_{r+1}^{(n+2)}}{e_r^{(n)}} = \frac{(n + 2)(r - \alpha)}{n(r + 1)(n - r + \alpha + 1)(n - r - 1)},$$

which is clearly less than unity, and the proof is complete.

2. Rearrangement of the Chebyshev Series

In this section we adopt for convenience the notation

$$T_n(x) = \sum_{k=0}^{n} a_k^{(n)} C_k^{\alpha}(x), \qquad \text{where} \quad C_k^{\alpha}(1) = 1. \tag{1}$$

Let

$$f_n(x) = \sum_{k=0}^{n} d_k^{(n)} T_k(x) + \sum_{k=n+1}^{\infty} d_k^{(n)} T_k(x) = p_n(x) + q(x)$$

Then $(L_n^0 \hat{f}_n)(x) = p_n(x)$ and $(L_n^\alpha \hat{f}_n)(x) \simeq p_n(x) + (L_n^\alpha q)(x)$, where formally

$$(L_n^\alpha q)(x) = \sum_{k=n+1}^{\infty} d_k^{(n)} \sum_{i=0}^{n} a_i^{(k)} C_k^\alpha(x)$$

$$= \sum_{r=0}^{n} b_r C_r^\alpha(x)$$

and we have

$$b_r = \sum_{j=1}^{\infty} a_r^{(n+j)} d_{n+j}^{(n)} \tag{2}$$

For $(L_n^\alpha \hat{f}_n)(1)$ to be well defined we require these expressions for the b_r to converge, when

$$(L_n^\alpha \hat{f}_n)(1) - (L_n^0 \hat{f}_n)(1) = \sum_{r=0}^{n} b_r . \tag{3}$$

3. Properties of the Coefficients $d_k^{(n)}$

Lemma 2. *The sums* $\sum_{j=1}^{\infty} d_{n+2j}^{(n)}$ *and* $\sum_{j=1}^{\infty} d_{n+2j-1}^{(n)}$ *both converge to non-positive limits.*

Proof. The Chebyshev series $\sum_{k=0}^{\infty} d_k^{(n)} T_k(x)$ clearly converges at $x = \pm 1$, i.e., $\sum_{k=0}^{\infty} d_k^{(n)}$ and $\sum_{k=0}^{\infty} (-1)^k d_k^{(n)}$ both converge, which is sufficient to ensure the convergence of the two sums indicated in our lemma.

In the sum $\sum_{k=n+1}^{\infty} d_k^{(n)}$ the coefficient $d_{[p[n+\frac{1}{2}]]} = 0$, if p is even, where $[x]$ signifies the integer part of $x \in \mathbb{R}$. Furthermore, since the sign of the $d_k^{(n)}$ is controlled by the sign of $\tan(\pi k/(2n+1))$, this coefficient $d_{[p(n+\frac{1}{2})]}^{(n)}$ has $n-1$ negative coefficients preceding it, and $n-1$ positive coefficients following it. Since $\tan(\pi k/(2n+1))$ is symmetrical about $[p(n+\frac{1}{2})]$ for p even and for $p(n+\frac{1}{2}) - n + 1 \leqslant k \leqslant p(n+\frac{1}{2}) + n - 1$ we have

$$| d_{[p(n+\frac{1}{2})]-r}^{(n)} | > | d_{[p(n+\frac{1}{2})]+r}^{(n)} | \qquad 1 \leqslant r \leqslant n-1.$$

Now for any even value of $p \geqslant 2$ sums of the form

$$A_p = \sum_{\substack{1-n \leqslant r \leqslant n-1 \\ r\,\text{even}}} d_{p(n+\frac{1}{2})+r}^{(n)}, \qquad B_p = \sum_{\substack{1-n \leqslant r \leqslant n-1 \\ r\,\text{odd}}} d_{p(n+\frac{1}{2})+r}^{(n)}$$

must be negative, since each positive $d_{p(n+\frac{1}{2})+r}^{(n)}$, $0 \leqslant r \leqslant n-1$ has a corresponding negative $d_{p(n+\frac{1}{2})-r}^{(n)}$ of greater modulus. Finally, both $\sum_{j=1}^{\infty} d_{n+2j}^{(n)}$ and $\sum_{j=1}^{\infty} d_{n+2j-1}^{(n)}$ consist of either sums of the form A_p or B_p and the proof is complete.

4. Norms of Projections

THEOREM. *With the* L_n^α *defined as before we have*

$$\min_{0 \leqslant \alpha \leqslant 1} \| L_n^\alpha \| = \| L_n^0 \| \qquad \forall n \geqslant 1.$$

Proof. From (3) we have

$$(L_n^\alpha \hat{f}_n)(1) - (L_n^0 \hat{f})_n(1) = \sum_{r=0}^{n} b_r ,$$

where $b_r = \sum_{j=1}^{\infty} a_r^{(n+j)} d_{n+j}^{(n)}$ from (2). In fact, since alternate $a_r^{(n+j)}$ are zero we have

$$b_r = \sum_{j=1}^{\infty} a_r^{(n+2j)} d_{n+2j}^{(n)} \qquad n + r \text{ even},$$

$$= \sum_{j=1}^{\infty} a_r^{(n+2j-1)} d_{n+2j-1}^{(n)} \qquad n + r \text{ odd}.$$

Since the $a_r^{(n+2j)}$ and $a_r^{(n+2j-1)}$ are negative and decrease in modulus (Lemma 1) we have, by application of Lemma 2 that $b_r > 0$ for $0 \leqslant r \leqslant n$ and the result follows from the inequalities for $0 < \alpha < 1$:

$$\| L_n^\alpha \| > \| L_n^\alpha \hat{f}_n \| > (L_n^\alpha \hat{f}_n)(1) > (L_n^0 \hat{f}_n)(1) = \| L_n^0 \| .$$

The case for $\alpha = 1$ can be deduced from the continuity of $\| L_n^\alpha \|$.

5. Remarks

It does not seem possible to extend this method of proof to the cases where $\alpha > 1$, since the sign pattern of Lemma 1 is no longer preserved, although computational results indicate that the result holds good for $\alpha > 1$. The case for $\alpha = 1$ can also be obtained by using the relation $T_n(x) = \frac{1}{2}\{U_n(x) - U_{n-2}(x)\}$, when the proof is especially straightforward.

Similar relationships between C_n^α, $C_n^{\alpha+1}$ may perhaps yield the result for $\alpha > 1$. Clearly, the existing theorem can be extended to show that the Chebyshev projection is minimal in any family F whose orthogonal elements can be arranged so that Lemma 1 continues to hold.

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